The value of incoming message multiplicities in distributed computing

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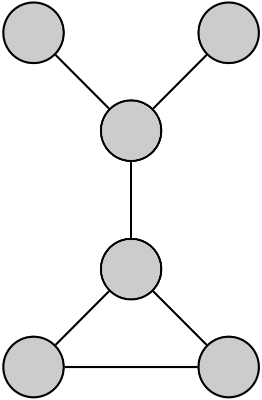
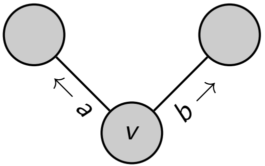
Distributed system



A graph, whose each node

- runs the same algorithm,
- can be given a local input,
- can communicate with its neighbours,
- produces a local output.

Communications happens in synchronous rounds



In every round, each node v

1. sends messages to its neighbours,
2. receives messages from its neighbours,
3. updates its state.

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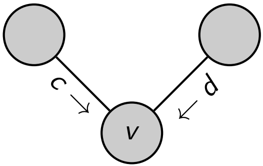

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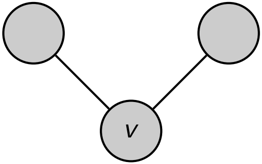

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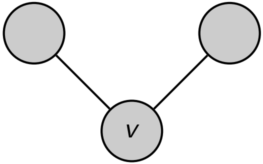

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1. sends messages to its neighbours,
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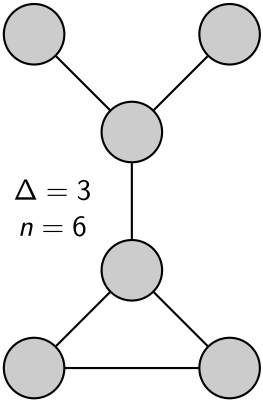
3. updates its state.

After the final round, each node announces its own output.

Focus on communication, not computation



The running time of an algorithm is the *number of communications rounds*.

The running time may depend on

- the maximum degree of the graph, Δ,
- the number of nodes, n.

Port numbering

A *port* of a graph $G = (V, E)$ is a pair (v, i), where $v \in V$ and $i \in \{1, 2, \dots, \deg(v)\}$. Let $P(G)$ be the set of all ports of G. A *port numbering* of G is a bijection $p\colon P(G) \to P(G)$ such that

$$p(v, i) = (u, j) \text{ for some } i \text{ and } j \quad \text{if and only if} \quad \{v, u\} \in E.$$

Intuitively, if $p(v, i) = (u, j)$, then (v, i) is an output port of node v that is connected to an input port (u, j) of node u.

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Intuitively, if $p(v, i) = (u, j)$, then (v, i) is an output port of node v that is connected to an input port (u, j) of node u.

We say that a port numbering p is *consistent* if we have

$$p(p(v, i)) = (v, i) \text{ for all } (v, i) \in P(G),$$

or, in other words, if the input port and the output port connected to the same neighbour always have the same number.

Graph classes and local inputs

For each positive integer Δ, denote by $\mathcal{F}(\Delta)$ the class of all simple undirected graphs of maximum degree at most Δ.

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An *input* for a graph $G = (V, E)$ is a function $f \colon V \to X$, where $X \ni \emptyset$ is a finite set. For each $v \in V$, the value $f(v)$ is called the *local input* of v.

The symbol $\emptyset \in X$ is used to indicate "no input".

Algorithms as state machines

Let $\Delta \in \mathbb{N}_+$ and let X be a set of local inputs. A *distributed state machine* for $(\mathcal{F}(\Delta), X)$ is a tuple $\mathcal{A} = (Y, Z, \sigma_0, M, \mu, \sigma)$, where

- Y is a set of states,
- $Z \subseteq Y$ is a finite set of stopping states,
- $\sigma_0 \colon \{0, 1, \ldots, \Delta\} \times X \to Y$ is a function that defines the initial state,
- M is a set of messages such that $\epsilon \in M$,
- $\mu \colon Y \times [\Delta] \to M$ is a function that constructs the outgoing messages, such that $\mu(z, i) = \epsilon$ for all $z \in Z$ and $i \in [\Delta]$,
- $\sigma \colon Y \times M^\Delta \to Y$ is a function that defines the state transitions, such that $\sigma(z, \overline{m}) = z$ for all $z \in Z$ and $\overline{m} \in M^\Delta$.

The special symbol $\epsilon \in M$ indicates "no message".

Execution

Let $G = (V, E) \in \mathcal{F}(\Delta)$, let p be a port numbering of G, let $f\colon V \to X$, and let $\mathcal{A}$ be a distributed state machine for $(\mathcal{F}(\Delta), X)$.

The state of the system in round $r \in \mathbb{N}$ is a function $x_r\colon V \to Y$, where $x_r(v)$ is the *state* of node v in round r. To initialise the nodes, set

$$x_0(v) = \sigma_0(\deg(v), f(v)) \quad \text{for each } v \in V.$$

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Then, assume that x_r is defined for some $r \in \mathbb{N}$. Let $(u, j) \in P(G)$ and $(v, i) = p(u, j)$. Now, node v receives the message

$$a_{r+1}(v, i) = \mu(x_r(u), j)$$

from its port (v, i) in round $r + 1$. For each $v \in V$, we define

$$\overline{a}_{r+1}(v) = (a_{r+1}(v, 1), a_{r+1}(v, 2), \ldots, a_{r+1}(v, \deg(v)), \epsilon, \epsilon, \ldots, \epsilon) \in M^{\Delta}.$$

Now we can define the new state of each node $v \in V$ as follows:

$$x_{r+1}(v) = \sigma(x_r(v), \overline{a}_{r+1}(v)).$$

Running time

Let $t \in \mathbb{N}$. If $x_t(v) \in Z$ for all $v \in V$, we say that $\mathcal{A}$ *stops in time* t in (G, f, p).

The *running time* of $\mathcal{A}$ in (G, f, p) is the smallest t for which this holds.

If $\mathcal{A}$ stops in time t in (G, f, p), the *output* of $\mathcal{A}$ in (G, f, p) is $x_t \colon V \to Y$.

For each $v \in V$, the *local output* of v is $x_t(v)$.

Graph problems

We study *graph problems* where

- problem instance is the communication graph (and the possible local inputs),
- the local outputs together define a solution.

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- problem instance is the communication graph (and the possible local inputs),
- the local outputs together define a solution.

Let X and Y be finite nonempty sets.

A *graph problem* is a function $\Pi_{X,Y}$ that maps each undirected simple graph $G = (V, E)$ and each input $f\colon V \to X$ to a set $\Pi_{X,Y}(G, f)$ of solutions.

Each *solution* $S \in \Pi_{X,Y}(G, f)$ is a function $S\colon V \to Y$.

Solving a graph problem

Let $\Pi_{X,Y}$ be a graph problem, $T\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and $\mathbf{A} = (\mathcal{A}_1, \mathcal{A}_2, \dots)$ such that each $\mathcal{A}_\Delta$ is a distributed state machine for $(\mathcal{F}(\Delta), X)$. Algorithm $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in time* T if the following holds for all $\Delta \in \mathbb{N}$, all finite graphs $G = (V, E) \in \mathcal{F}(\Delta)$, all inputs $f\colon V \to X$ and all port numberings p of G:

1. $\mathcal{A}_\Delta$ stops in time $T(\Delta, |V|)$ in (G, f, p).
2. The output of $\mathcal{A}_\Delta$ in (G, f, p) is in $\Pi_{X,Y}(G, f)$.

Solving a graph problem

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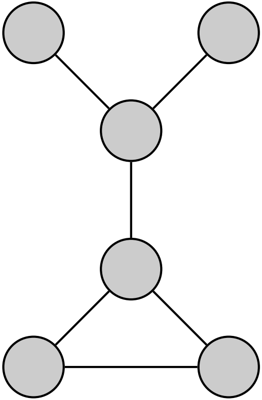
We say that $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in time* T *assuming consistency* if the above holds for all consistent port numberings p of G.

If $T(\Delta, n)$ does not depend on n, we say that $\mathbf{A}$ *solves* $\Pi_{X,Y}$ *in constant time* or that $\mathbf{A}$ *is a local algorithm for* $\Pi_{X,Y}$.

Graph problems



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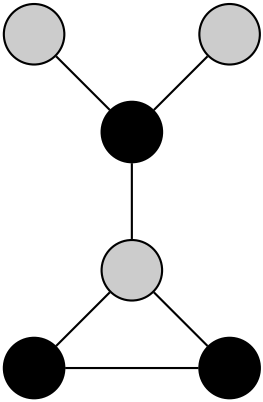

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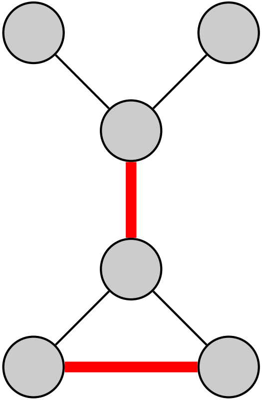
Example problems:

- minimum vertex cover,

Graph problems



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Example problems:

- minimum vertex cover,
- maximal matching.

Variants of the model of computation

$\mathcal{VV}$ is the class of all distributed state machines (send a vector, receive a vector).

We can place different restrictions on the algorithms:

- $\mathcal{VB}$: broadcast the same message to all neighbours:

$$\mu(y, i) = \mu(y, j) \text{ for all } i, j \in \{1, 2, \dots, \Delta\} \text{ and } y \in Y,$$

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- $\mathcal{MV}$: receive a multiset of messages:

  $$\text{multiset}(\overline{a}) = \text{multiset}(\overline{b}) \Rightarrow \sigma(y, \overline{a}) = \sigma(y, \overline{b}) \text{ for all } y \in Y,$$

- $\mathcal{SV}$: receive a set of messages:

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- $\mathcal{MB} = \mathcal{MV} \cap \mathcal{VB}$ and $\mathcal{SB} = \mathcal{SV} \cap \mathcal{VB}$.

Variants of the model of computation

$$\begin{aligned}
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\end{aligned}$$

Complexity classes

Let P be the class of all graph problems.

$$
\begin{aligned}
\mathsf{VV_c} &= \{\Pi \in \mathsf{P} : \text{there is } \mathbf{A} \in \mathbf{VV} \text{ that solves } \Pi \text{ assuming consistency}\},\\
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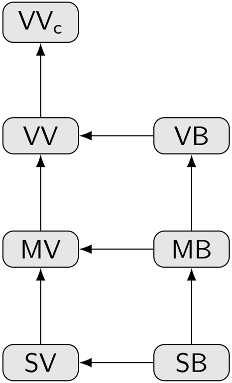
\end{aligned}
$$

Containment relations between the classes



Trivial relations:

- $SV \subseteq MV \subseteq VV \subseteq VV_c$,
- $SB \subseteq MB \subseteq VB$,
- $VB \subseteq VV$,
- $MB \subseteq MV$,
- $SB \subseteq SV$.

Non-trivial: $SV \subseteq VB$? $VB \subseteq SV$?

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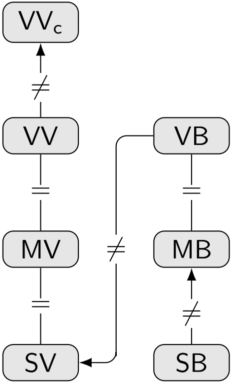

Hella, Järvisalo, Kuusisto, Laurinharju, L., Luosto, Suomela, Virtema (PODC 2012):

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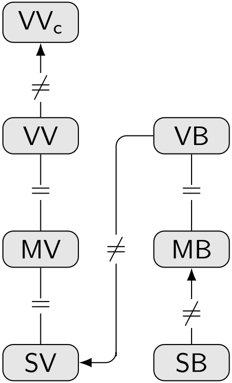

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Hella et al. also showed that constant-time variants of the classes can be characterised by certain modal logics.

The relationship of MV and SV

Trivially SV $\subseteq$ MV.

Hella, Järvisalo, Kuusisto, Laurinharju, L., Luosto, Suomela, Virtema (PODC 2012):

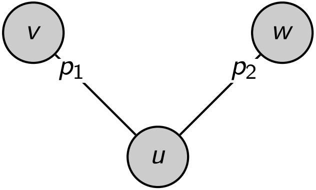
Theorem

Let Π be a graph problem and let $T : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Assume that there is an algorithm $\mathbf{A} \in \mathbf{MV}$ that solves Π in time T. Then there is an algorithm $\mathbf{B} \in \mathbf{SV}$ that solves Π in time T', where $T'(n, \Delta) = T(n, \Delta) + 2\Delta - 2$.

It follows that SV $=$ MV.

Idea behind the simulation theorem

First, solve the following *simulation problem* by an $\mathcal{SV}$-algorithm:



If $p_1 = p_2$, then
output(v) $\neq$ output(w).

Now the pair

(output, port number)

is distinct for each neighbour.

This takes $2\Delta - 2$ communication rounds.

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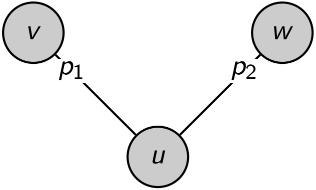

If $p_1 = p_2$, then
$\text{output}(v) \neq \text{output}(w)$.

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This takes $2\Delta - 2$ communication rounds.

Then, simulate the $\mathcal{MV}$-algorithm by attaching the above pair to each message. That way we can reconstruct the message multiplicities.

New results: Lower bounds for the simulation

Is the overhead of $2\Delta - 2$ rounds really needed to reconstruct the message multiplicities by an $\mathcal{SV}$-algorithm?

Theorem

For each $\Delta \geq 2$ there is a graph $G = (V, E) \in \mathcal{F}(\Delta)$, a port numbering p of G and nodes $v, u, w \in V$ such that when executing any algorithm $\mathcal{A} \in \mathcal{SV}$ in (G, p), node v receives identical messages from its neighbours u and w in rounds $1, 2, \ldots, 2\Delta - 2$.

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Theorem

There is a graph problem Π that can be solved in one round by an algorithm in **MV** *but that requires at least time T, where $T(n, \Delta) \geq \Delta$ for all $\Delta \geq 2$, when solved by an algorithm in* **SV**.

Notation for outgoing port numbers

If $p(v, i) = (u, j)$, we write $\pi(v, u) = i$. That is, $\pi(v, u)$ is the number of the output port of v that is connected to u.

How to prove that two nodes stay in the same state?

Definition

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, and let p and p' be port numberings of G and G', respectively. An *r-$\mathcal{SV}$-bisimulation* between nodes $v \in V$ and $v' \in V'$ is a sequence of binary relations $B_r \subseteq B_{r-1} \subseteq \cdots \subseteq B_0 \subseteq V \times V'$ such that the following conditions hold for $1 \leq i \leq r$:

1. $(v, v') \in B_r$.
2. If $(u, u') \in B_0$, then $\deg_G(u) = \deg_{G'}(u')$ and $f(u) = f'(u')$.
3. If $(u, u') \in B_i$ and $\{u, w\} \in E$, then there is $w' \in V'$ such that $\{u', w'\} \in E'$, $(w, w') \in B_{i-1}$ and $\pi(w, u) = \pi'(w', u')$.
4. If $(u, u') \in B_i$ and $\{u', w'\} \in E'$, then there is $w \in V$ such that $\{u, w\} \in E$, $(w, w') \in B_{i-1}$ and $\pi(w, u) = \pi'(w', u')$.

How to prove that two nodes stay in the same state?

We say that $v \in V$ and $v' \in V'$ are *r-$\mathcal{SV}$-bisimilar* and write $(G, f, v, p) \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} (G', f', v', p')$ (or simply $v \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} v'$) if there exists an *$r$-$\mathcal{SV}$-bisimulation* between them.

Lemma

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, and let p and p' be port numberings of G and G', respectively. If $(G, f, v, p) \mathrel{\underline{\leftrightarrow}}_r^{\mathcal{SV}} (G', f', v', p')$ for some $r \in \mathbb{N}$, $v \in V$ and $v' \in V'$, then for all algorithms $\mathcal{A} \in \mathcal{SV}$ we have $x_t(v) = x'_t(v')$ for all $t = 0, 1, \dots, r$, that is, the state of v and v' is identical in rounds $0, 1, \dots, r$.

Bisimilarity

Lemma

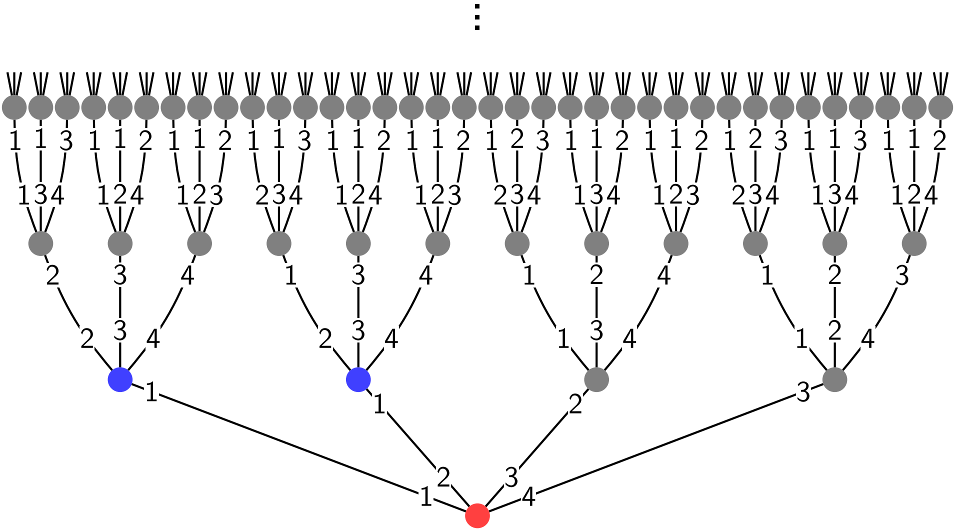
The r-$\mathcal{SV}$-bisimilarity relation $\underline{\leftrightarrow}_r^{\mathcal{SV}}$ is an equivalence relation in the class of quadruples (G, f, v, p), where $G = (V, E)$ is a graph, f is an input for G, p is a port numbering of G and $v \in V$.

Lemma

Let $G = (V, E)$ and $G' = (V', E')$ be graphs, let f and f' be inputs for G and G', respectively, let p and p' be port numberings of G and G', respectively, and let $v \in V$, $v' \in V'$. Then $(G, f, v, p) \underline{\leftrightarrow}_r^{\mathcal{SV}} (G', f', v', p')$ iff the following conditions hold:

1. $(G, f, v, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', v', p')$.
2. *If* $\{v, w\} \in E$, *then there is* $w' \in V'$ *such that* $\{v', w'\} \in E'$, $(G, f, w, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', w', p')$ *and* $\pi(w, v) = \pi'(w', v')$.
3. *If* $\{v', w'\} \in E'$, *then there is* $w \in V$ *such that* $\{v, w\} \in E$, $(G, f, w, p) \underline{\leftrightarrow}_{r-1}^{\mathcal{SV}} (G', f', w', p')$ *and* $\pi(w, v) = \pi'(w', v')$.

The lower bound construction G_d for $d = 4$



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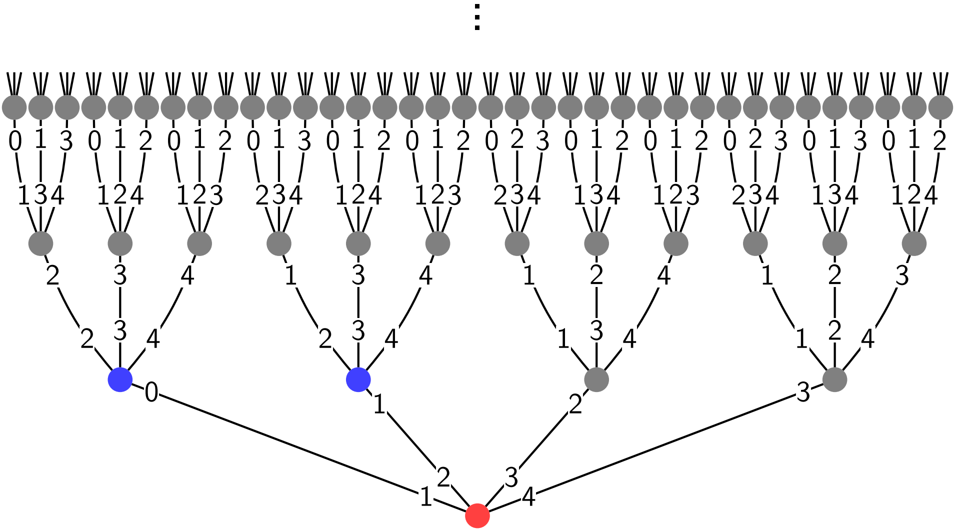


Definition of the graph G_d

1. $\emptyset \in V_d$.
2. $((1,0)), ((2,1)), ((3,2)), ((4,3)), \ldots, ((d, d-1)) \in V_d$.
3. If $(a_1, a_2, \ldots, a_i) \in V_d$, where i is odd and $i < 2d$, then $(a_1, a_2, \ldots, a^j_{i+1}) \in V_d$ for all $j = 1, 2, \ldots, d-1$, where $a^j_{i+1} = (c^j_1, c^j_2)$ is defined as follows. Let $(b_1, b_2) = a_i$ and $b^+_2 = 1$ if $b_2 = 0$, $b^+_2 = b_2$ otherwise. Define
$$c^j_1 = \min(\{1, 2, \ldots, d\} \setminus \{b^+_2, c^1_1, c^2_1, \ldots, c^{j-1}_1\}),$$
$$c^j_2 = \min(\{1, 2, \ldots, d\} \setminus \{b_1, c^1_2, c^2_2, \ldots, c^{j-1}_2\}).$$
4. If $(a_1, a_2, \ldots, a_i) \in V_d$, where i is even and $0 < i < 2d$, then $(a_1, a_2, \ldots, a^j_{i+1}) \in V_d$ for all $j = 1, 2, \ldots, d-1$, where $a^j_{i+1} = (c^j_1, c^j_2)$ is defined as follows. Let $(b_1, b_2) = a_i$. Define
$$c^j_1 = \min(\{1, 2, \ldots, d\} \setminus \{b_2, c^1_1, c^2_1, \ldots, c^{j-1}_1\}),$$
$$c^j_2 = \min(\{0, 1, \ldots, d-1\} \setminus \{b_1, c^1_2, c^2_2, \ldots, c^{j-1}_2\}).$$

Definition of the graph G_d

The set E_d of edges consists of all pairs $\{v, u\}$, where $v = (a_1, a_2, \ldots, a_i) \in V_d$ and $u = (a_1, a_2, \ldots, a_i, a_{i+1}) \in V_d$ for some $i \in \{0, 1, \ldots\}$.

If $v = (a_1, a_2, \ldots, a_i)$ and $u = (a_1, a_2, \ldots, a_{i+1})$, where $a_{i+1} = (b_1, b_2)$, the outgoing port number from v to u is $\pi_d(v, u) = b_1$ and the outgoing port number from u to v is $\pi_d(u, v) = b_2$.

Pairs of separating walks (PSWs)

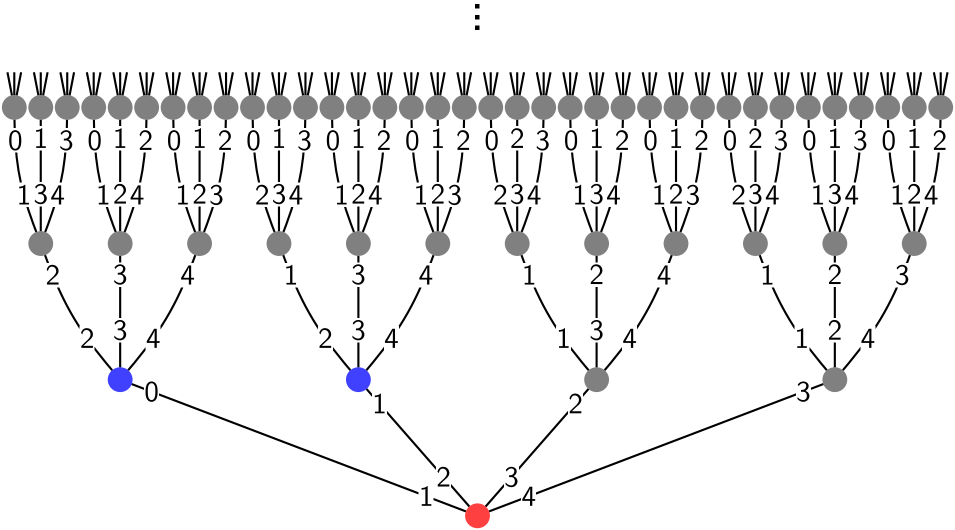
A *walk* is a sequence $\overline{v} = (v_0, v_1, \ldots, v_k)$ of nodes such that $\{v_i, v_{i+1}\} \in E_d$ for all $i = 0, 1, \ldots, k-1$.

A pair $(\overline{v}_1, \overline{v}_2)$ of walks, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for all $i = 1, 2$, is called a *pair of separating walks (PSW) of length k in G_d* if the following conditions hold:

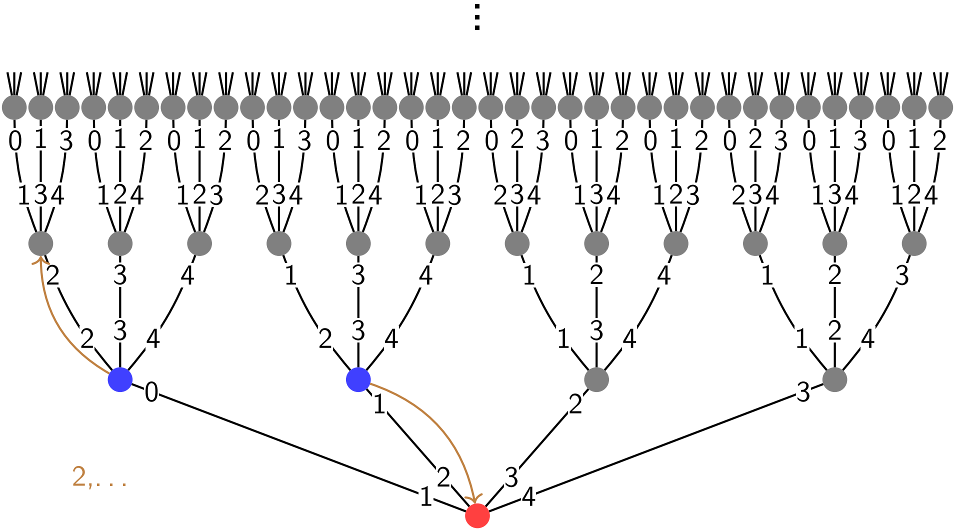
1. $v_0^1 = ((1,0))$ and $v_0^2 = ((2,1))$.
2. $\pi_d(v_j^1, v_{j-1}^1) = \pi_d(v_j^2, v_{j-1}^2)$ for all $j = 1, 2, \ldots, k$.
3. There is $v_{k+1}^1 \in V_d$ with $\{v_k^1, v_{k+1}^1\} \in E_d$ such that there is no $v_{k+1}^2 \in V_d$ for which $\{v_k^2, v_{k+1}^2\} \in E_d$ and $\pi_d(v_{k+1}^1, v_k^1) = \pi_d(v_{k+1}^2, v_k^2)$.

We say that a pair of separating walks of length k in G_d is *critical* if there does not exist a pair of separating walks of length k' in G_d for any $k' < k$.

A pair of separating walks in G_4

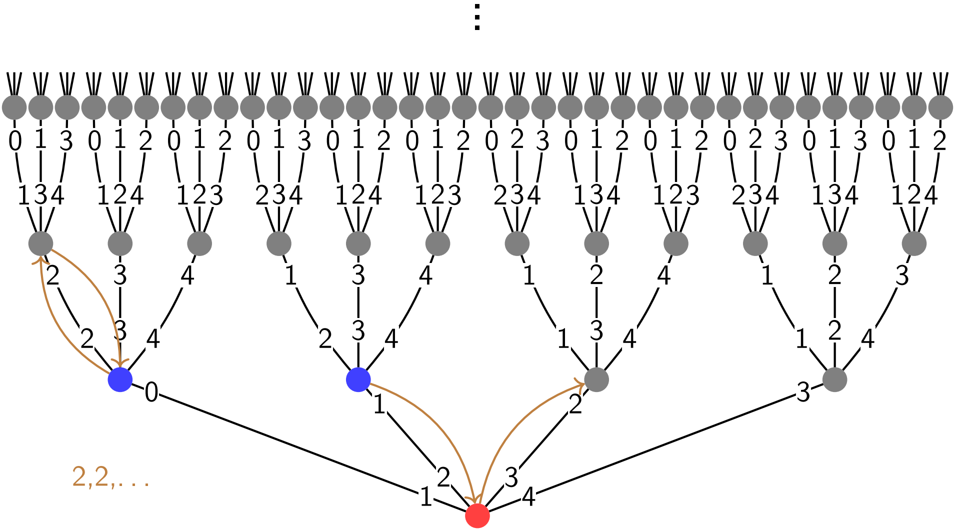


A pair of separating walks in G_4

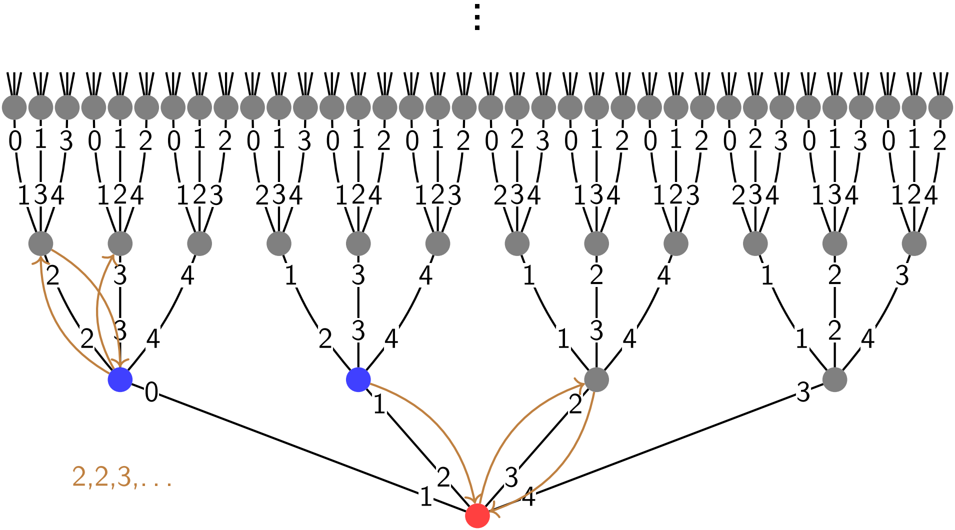


2,. . .

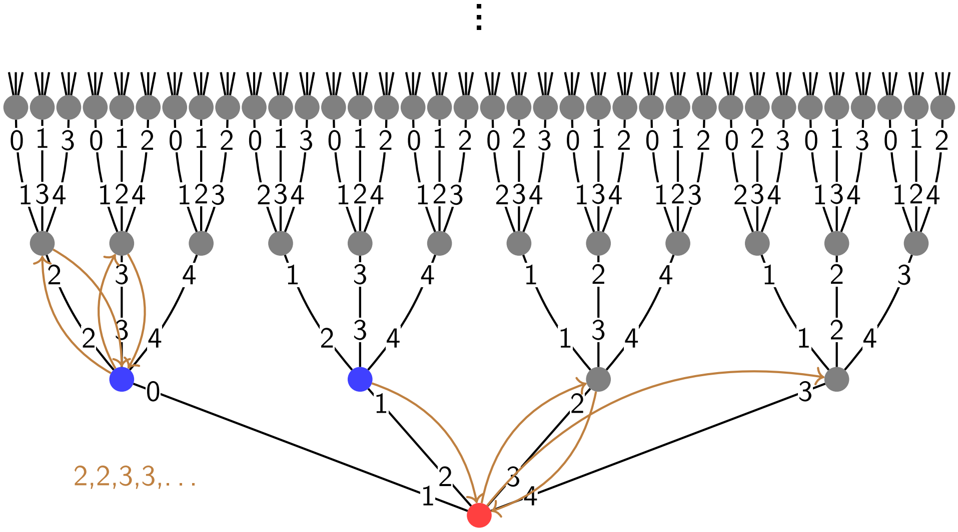
A pair of separating walks in G_4



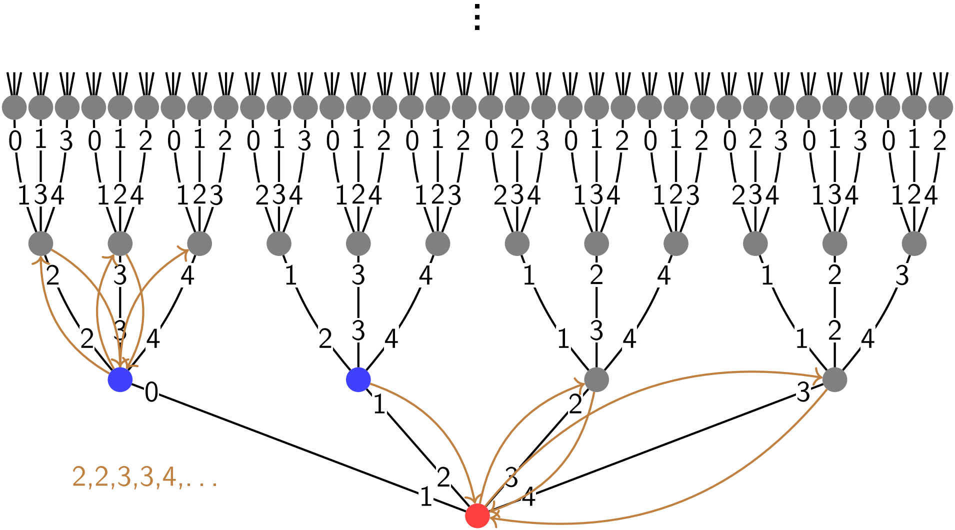
A pair of separating walks in G_4



A pair of separating walks in G_4

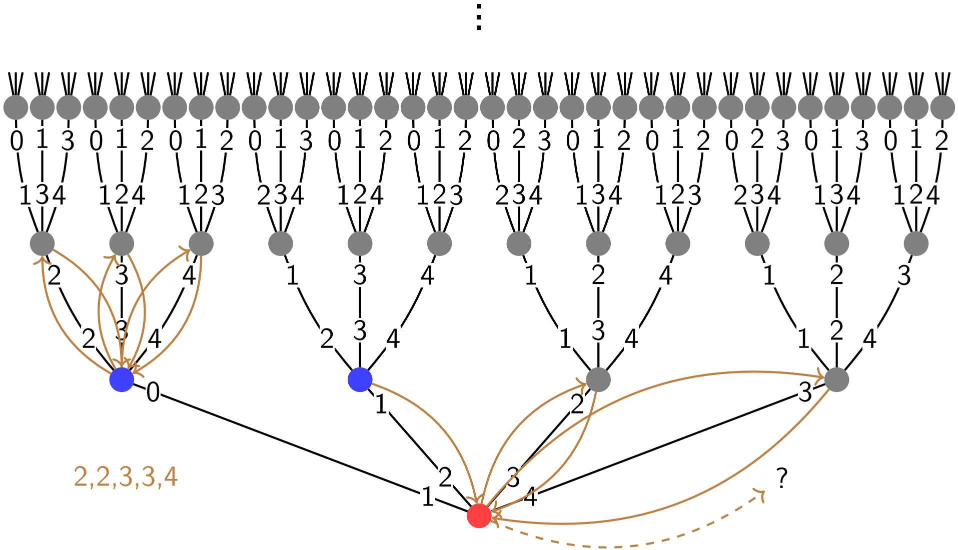


A pair of separating walks in G_4



2,2,3,3,4,. . .

A pair of separating walks in G_4



Proof idea

1. The length of a critical PSW in G_d is at least $2d - 3$.
   1. The sequence of port numbers starts from 1 or 2.
   2. The numbers grow slowly along the walks.
   3. Eventually the sequence reaches d.
2. If k is the largest integer for which $((1,0)) \underline{\leftrightarrow}_k^{\mathcal{SV}} ((2,1))$, then there is a PSW of length k in G_d.

Definitions

If $v = (a_1, a_2, \ldots, a_i)$ and $u = (a_1, a_2, \ldots, a_{i+1})$, we say that node v is the *parent* of node u and that u is a *child* of v.

We say that the node v is *even* if i is even and *odd* if i is odd.

If $a_i = (b_1, b_2)$, we call (b_1, b_2) the *type* of node v.

Easy observations

Lemma

For each d, we have $\deg(v) \in \{1, d\}$ for all $v \in V_d$, and thus $G_d \in \mathcal{F}(d)$. Additionally, G_d is a subgraph of G_{d+1}.

Easy observations

Lemma

For each d, we have $\deg(v) \in \{1, d\}$ for all $v \in V_d$, and thus $G_d \in \mathcal{F}(d)$. Additionally, G_d is a subgraph of G_{d+1}.

Lemma

Let $v \in V_d$ and $a \in \{0, 1, \ldots, d\}$. Then there is at most one node $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$.

Easy observations

Lemma

Let $v = (a_1, a_2, \ldots, a_i) \in V_d$, where $i < 2d$. If v is odd, then for all $a \in \{1, 2, \ldots, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. If v is even, then either for all $a \in \{0, 1, \ldots, d-1\}$ or for all $a \in \{0, 1, \ldots, d-2, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. In the case of even v and $a = d$, node u is the parent of node v.

Easy observations

Lemma

Let $v = (a_1, a_2, \ldots, a_i) \in V_d$, where $i < 2d$. If v is odd, then for all $a \in \{1, 2, \ldots, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. If v is even, then either for all $a \in \{0, 1, \ldots, d-1\}$ or for all $a \in \{0, 1, \ldots, d-2, d\}$ there exists $u \in V_d$ such that $\{v, u\} \in E_d$ and $\pi_d(u, v) = a$. In the case of even v and $a = d$, node u is the parent of node v.

Lemma

Let $\{v, u\} \in E_{d+1} \setminus E_d$ be such that $v \in V_d$. Then u is a child of v. If v is odd, then $\pi_{d+1}(v, u) = \pi_{d+1}(u, v) = d + 1$. If v is even, then $\pi_{d+1}(v, u) = d + 1$ and $\pi_{d+1}(u, v) \in \{d - 1, d\}$.

Walks in isomorphic subtrees

Lemma

Let $(\overline{v}_1, \overline{v}_2)$, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for some $k \leq 2d-3$ and all $i = 1, 2$, be a PSW in G_d. If for some $\ell \in \{0, 1, \ldots, k-1\}$ the node $v_{\ell+1}^i$ is a child of node v_ℓ^i for all $i = 1, 2$, and we have $\pi_d(v_\ell^1, v_{\ell+1}^1) = \pi_d(v_\ell^2, v_{\ell+1}^2)$, then $(\overline{v}_1, \overline{v}_2)$ is not a critical PSW in G_d.

Extending a PSW

Lemma

Let $(\overline{v}_1, \overline{v}_2)$ be a PSW of length $k \leq 2d - 3$ in G_d. Then there is a PSW of length $k + 2$ in G_{d+1}.

Second-to-last node is in $V_d \setminus V_{d-1}$

Lemma

Let $(\overline{v}_1, \overline{v}_2)$*, where* $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ *for some* $k \leq 2d-3$ *and all* $i = 1, 2$*, be a critical PSW in* G_d*. Then we have* $v_{k-1}^i \in V_d \setminus V_{d-1}$ *for some* $i \in \{1, 2\}$*.*

Pair of walks that is not a PSW

Lemma

Let $(\overline{v}_1, \overline{v}_2)$, where $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ for some $k \leq 2d-3$ and all $i = 1, 2$, be a pair of walks in G_d such that conditions (1) and (2) hold. If $(\overline{v}_1, \overline{v}_2)$ is not a PSW in G_d, then for each neighbour $v_{k+1}^1 \in V_d$ of v_k^1 there is a neighbour $v_{k+1}^2 \in V_d$ of v_k^2 such that $\pi_d(v_{k+1}^1, v_k^1) = \pi_d(v_{k+1}^2, v_k^2)$, and vice versa.

The main lemma

Lemma

Let $(\overline{v}_1, \overline{v}_2)$*, where* $\overline{v}_i = (v_0^i, v_1^i, \ldots, v_k^i)$ *for some* $k \leq 2d - 3$ *and all* $i = 1, 2$*, be a critical PSW in* G_d*. Then* $(\overline{v}'_1, \overline{v}'_2)$*, where* $\overline{v}'_i = (v_0^i, v_1^i, \ldots, v_{k-2}^i)$ *for all* $i = 1, 2$*, is a PSW in* G_{d-1}*.*

Minimum length of a PSW and bisimilarity

Lemma

Let $(\overline{v}_1, \overline{v}_2)$ *be a PSW of length* $k \leq 2d - 3$ *in* G_d*. Then* $k \geq 2d - 3$*.*

Minimum length of a PSW and bisimilarity

Lemma

Let $(\overline{v}_1, \overline{v}_2)$ be a PSW of length $k \leq 2d - 3$ in G_d. Then $k \geq 2d - 3$.

Lemma

We have $((1,0)) \underline{\leftrightarrow}^{\mathcal{SV}}_{2d-3} ((2,1))$, that is, the nodes $((1,0))$ and $((2,1))$ of G_d are $(2d-3)$-$\mathcal{SV}$-bisimilar.

Conclusion

$\mathcal{MV}$: Send a vector, receive a multiset.

$\mathcal{SV}$: Send a vector, receive a set.

Previously:

- It is possible to simulate $\mathcal{MV}$ in $\mathcal{SV}$ by using $2\Delta - 2$ extra rounds.

This work:

- $2\Delta - 2$ rounds are necessary to solve the simulation problem.
- There is a graph problem for which the difference in running time between $\mathcal{MV}$ and $\mathcal{SV}$ is $\Delta - 1$ rounds.

The thesis *A Classification of Weak Models of Distributed Computing* is available at `http://hdl.handle.net/10138/144214`.